Recurrence relations and sequences.

1. Write down recurrence relations for each of the following sequences.

a) 10, 13, 16, 19, 22, . . . b) 2, 9, 16, 23, 30, . . .

c) 8, 9, 10, 11, 12, . . . d) 2, 4, 6, 8, 10, . . .

e) 5, 61/2, 8, 91/2, 11, . . . f) 7, 4, 1, -2, -5, . . .

g) 90, 80, 70, 60, 50, . . . h) 8, 16, 24, 32, 40, . . .

i) 5, 10, 20, 40, 80, . . . j) 1000, 200, 40, 8, 1.6, . . .

k) 25, 20, 16, 12.8, . . . l) 8, 12, 18, 27, 401/2, . . .

In each example write down a formula for u_n in terms of n.

2. Write down the first five terms in each of the following sequences

   a) $u_{n+1} = u_n - 3$, $u_1 = 3$

   b) $u_{n+1} = u_n - 1$, $u_1 = 7$

   c) $u_{n+1} = 4u_n$, $u_1 = 1/2$

   d) $u_{n+1} = 1/2\ u_n$, $u_1 = 2000$

   e) $u_{n+1} = 2u_n - 3$, $u_1 = 4$

   f) $u_{n+1} = 3u_n - 1$, $u_1 = 1$

   g) $u_{n+1} = {}^3/_4\ u_n$, $u_1 = 256$

3. In each example find the first few terms and decide if a limit exists.

   If it does, find it.

   a) $u_{n+1} = 1/2\ u_n + 4$, $u_1 = 6$

   b) $u_{n+1} = ¼\ u_n + 24$, $u_1 = 16$

   c) $u_{n+1} = 2u_n + 1$, $u_1 = 4$

   d) $u_{n+1} = 0.6u_n + 20$, $u_1 = 10$

   e) $u_{n+1} = 0.9u_n + 3$, $u_1 = 40$

4. $u_n = 0.5n(n + 2)$ Find u_2 and u_7 and find n if $u_n = 24$.

5. $u_1 = -1$, $u_2 = 3$, $u_3 = 7$ Write down a recurrence relation for the sequence and find a formula for u_n .

6. $u_1 = 2$, $u_2 = 6$, $u_3 = 18$ Write down a recurrence relation for the sequence and find a formula for u_n .

7. If $s_n = 1/4; n(n + 1)$, find u_1, u_2 and u_3 . Find a recurrence relation for the sequence and a formula for the u_n .

8. $u_n = 1/2\ n\{1 - (-1)^n\}$ Find u_{15} and u_{16}

   List the first six terms of the sequence.

12. S_n is the sum of the first n terms of the sequence

13. A sequence is defined by $u_n = 4n + 7$. Write down u_1, u_2, and u_3.

Evaluate $u_{n+1} - u_n$.

14. A sequence is defined by $u_{n+1} = 0.2u_n + 4$, $u_1 = 3$.

15. If $S_n = n^2 + 4n$, evaluate S_1, S_2 and S_3 and hence find the first three terms of the sequence.

16. Find the first three terms of the sequence in which sum of the first n terms is

$S_n = 3n^2 + 9n$.

Find a formula for u_n .

17. Find the fourth term of the sequence where the sum of the first n terms is

$S_n = n^3 - 2n$.

18. Find the tenth term of the sequence where the sum of the first n terms is

$S_n = 2^n + 1$.

19. A sequence is given by $u_{n+1} = 2u_n + 1$

Find expressions for u_1, u_2, and u_3 showing that $u_3 = 2^3.u_0 + 2^2 + 2 + 1$

Deduce a similar expression for u_n .

20. A sequence is given by $u_{n+1} = 5u_n + 1$

Find expressions for u_1, u_2, and u_3 showing that $u_3 = 5^3.u_0 + 5^2 + 5 + 1$

Deduce a similar expression for u_n .

21. A sequence is defined by $u_{n+1} = a.u_n + b$.

If $u_1 = 5$, $u_2 = 7$ and $u_3 = 13$ find a and b .

22. A sequence is given by $u_{n+1} = 5u_n + b$ where b is a constant.

If $u_1 = 8$ and $u_3 = 224$, find b .

23. Find the limit of the sequence $u_n = {}^3/_4\, u_n + 5$ with $u_1 = 12$.

24. Find the limit of the sequence $u_{n+1} = 0.8u_n + 14$

25. If $u_n = 2^n + 3$, list the first 5 terms. Find a and b such that $u_{n+1} = a.u_n + b$.

Recurrence relations and sequences. Solutions

	Recurrence relation	Formula
1. a)	$u_{n+1} = u_n + 3$, $u_1 = 10$	$u_n = 3n + 7$
b)	$u_{n+1} = u_n + 7$, $u_1 = 2$	$u_n = 7n - 5$
c)	$u_{n+1} = u_n + 1$, $u_1 = 8$	$u_n = n + 7$
d)	$u_{n+1} = u_n + 2$, $u_1 = 2$	$u_n = 2n$
e)	$u_{n+1} = u_n + 1.5$, $u_1 = 5$	$u_n = {}^3/_2\, n + {}^7/_2$
f)	$u_{n+1} = u_n - 3$, $u_1 = 7$	$u_n = 10 - 3n$
g)	$u_{n+1} = u_n - 10$, $u_1 = 90$	$u_n = 100 - 10n$
h)	$u_{n+1} = u_n + 8$, $u_1 = 10$	$u_n = 8n$
i)*	$u_{n+1} = 2u_n$, $u_1 = 5$	$u_n = 5 \,.\, 2^{n-1}$
j)*	$u_{n+1} = {}^1/_5\, u_n$, $u_1 = 1000$	$u_n = (1/5^{n-1}) \,.\, 1000$
k)*	$u_{n+1} = {}^4/_5\, u_n$, $u_1 = 25$	$u_n = 25 \,.\, 0.8^n$
l)*	$u_{n+1} = 1.5u_n$, $u_1 = 8$	$u_n = 8 \,.\, 1.5^{n-1}$

*{Note : * means not easy!}*

2. a) 8, 11, 14, 17, 10
   b) 7, 6, 5, 4, 3
   c) 1/2, 2, 8, 32, 128
   d) 2000, 1000, 500, 250, 125
   e) 4, 5, 7, 11, 19, 35
   f) 1, 2, 5, 14, 41
   g) 256, 192, 144, 108, 81
3. a) limit = 8 b) 32 c) no limit d) 50 e) 30
4. $u_2 = 4$, $u_7 = 31.5$ $1/2\, n(n + 2) = 24 \Rightarrow n = 6$ {or -8}
5. $u_{n+1} = u_n + 4$, $u_1 = -1$; $u_n = 4n - 5$
6. $u_1 = 2$, $u_2 = 6 = 2 \,.\, 3$, $u_3 = 18 = 2 \,.\, 3^2$,
   $u_4 = 54 = 2 \,.\, 3^3$, $u_5 = 162 = 2 \,.\, 3^4$
   $u_{n+1} = 3u_n$, $u_1 = 2$; $u_n = 2 \,.\, 3^n$
7. $S_1 = 1/2$, $S_2 = {}^3/_2$, $S_3 = 3$, $S_4 = 5$
   $u_1 = 1/2$, $u_2 = S_2 - S_1 = 1$, $u_3 = S_3 - S_2 = 11/2$, $u_4 = S_4 - S_3 = 2$
   $u_{n+1} = u_n + 0.5$, $u_1 = 0.5$; $u_n = 1/2\, n$
8. 1, 0, 3, 0, 5, 0 $u_{15} = 15$, $u_{16} = 0$ $u_n = n$ if n is odd

$u_n = 0$ if n is even

9. a) 1 b) 3

10. Let L be the limit then $L = 20 + {}^3/_4\, L$ => 1/4; L = 20 => L = 80

11. Let L be the limit then L = 4 - 0.1L => 1.1L = 4 => L = 4 ÷ 1.1 = 3.636363...

12. $S_1 = 1$, $S_2 = 1.5$, $S_3 = 1.75$, $S_4 = 1.875$, $S_5 = 1.9375$,

$S_6 = 1.96875$, $S_7 = 1.984375$, $S_8 = 1.992$ lim $S_n = 2$

13. $u_n = 4n + 7$ $u_1 = 11$, $u_2 = 15$, $u_3 = 19$ $u_{n+1} - u_n = 4$

14. Let L be the limit => L = 0.2L + 4 => 0.8L = 4 => L = 4 ÷ 0.8 = 5

15. $S_1 = 5$, $S_2 = 12$, $S_3 = 21$; $u_1 = 5$, $u_2 = 7$, $u_3 = 9$, $u_4 = 11$; $u_n = 2n + 3$

16. $S_1 = 12$, $S_2 = 30$, $S_3 = 54$; $u_1 = 12$, $u_2 = 18$, $u_3 = 24$; $u_n = 6n + 6$

17. $S_3 = 21$, $S_4 = 56$, $U_4 = S_4 - S_3 = 56 - 21 = 35$

18. $S_9 = 513$, $S_{10} = 1025$ $u_{10} = S_{10} - S_9 = 1025 - 513 = 512$

19. $u_{n+1} = 2u_n + 1$, $u_1 = 2u_0 + 1$,

$u_2 = 2u_1 + 1 = 2[2u_0 + 1] + 1 = 2^2.u_0 + 2 + 1$

$u_3 = 2u_2 + 1 = 2[2^2.u_0 + 2 + 1] + 1 = 2^3u_0 + 2^2 + 2 + 1$

Similarly, $u_4 = 2^4.u_0 + 2^3 + 2^2 + 2 + 1$

$u_n = 2^n.u_0 + 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$

20. $u_{n+1} = 5u_n + 1$, $u_1 = 5u_0 + 1$,

$u_2 = 5u_1 + 1 = 5[5u_0 + 1] + 1 = 5^2.u_0 + 5 + 1$

$u_3 = 5u_2 + 1 = 5[5^2.u_0+5+1]+1 = 5^3u_0 + 5^2 + 5 + 1$

Similarly, $u_4 = 5^4.u_0 + 5^3 + 5^2 + 5 + 1$

$u_n = 5^n.u_0 + 5^{n-1} + 5^{n-2} + 5^{n-3} + \ldots + 5 + 1$

21. $u_{n+1} = a.u_n + b$ $u_2 = a.u_1 + b$ $7 = 5a + b$ . . .(1)

$u_3 = a.u_2 + b$ $13 = 7a + b$ . . .(2)

=> a = 3, b = -8

22. $u_{n+1} = 5u_n + b$ $u_2 = 5u_1 + b$ $u_3 = 5u_2 + b = 5[5u_1 + b] + b = 25u_1 + 6b$

224 = 25 . 8 + 6b => 6b = 24 => b = 4

23. Let L be the limit $L = {}^3/_4\, L + 5$ => L = 20

24. L = 0.8L + 14 => 0.2L = 14 => L = 70

25. 5, 7, 11, 19, 35

Let $u_{n+1} = a.u_n + b$ and use with n = 1, 2

5a + b = 7 & 7a + b = 11 => a = 2, b = -3

=> $u_{n+1} = 2u_n - 3$, $u_1 = 5$

Pollution problems

1. The initial quantity of pollution in the loch is 25 tons, the Council remove 35% during the week and the factory discharges 8 tons into the loch each Sunday.

   i) Find the amount of pollution after 1, 2, 3 and 4 weeks

   ii) Establish a recurrence relation and hence find the long term state of the loch.

Repeat question 1 for each situation below.

2. Initial quantity of pollution = 33 tons

   Council remove 35% per week during the week

   Factory is allowed to discharge 20 tons per week at the end of the week

3. Initial quantity of pollution = 33 tons

   Council remove 60% per week during the week

   Factory is allowed to discharge 15 tons per week at the end of the week

4. Initial quantity of pollution = 16.5 tons

   Council remove 22% per week during the week

   Factory is allowed to discharge 6 tons per week at the end of the week

Pollution problems Solutions

A_0 = initial pollution L = long term state of the loch {limit of recurr. rel.}

1. $A_0 = 25$ $A_{n+1} = 0.73A_n + 20$ . . (1)

   First 4 weeks 24.25, 23.76, 23.45, 23.24

   Let L be the limit of (1) L = 0.65L + 8 => L = 8 ÷ 0.35 = 22.86 tons

2. $A_0 = 33$ $A_{n+1} = 0.65A_n + 8$ . . (1)

   First 4 weeks 44.1, 52.2, 58.1, 62.4

   Let L be the limit of (1) L = 0.73L + 20 => L = 20 ÷ 0.73 = 74.1 tons

3. $A_0 = 44$ $A_{n+1} = 0.4A_n + 15$ . . (1)

   First 4 weeks 32.6, 28.0, 26.2, 25.5

   Let L be the limit of (1) L = 0.4L + 15 => L = 15 ÷ 0.6 = 25 tons

4. $A_0 = 16.5$ $A_{n+1} = 0.78A_n + 6$ . . (1)

   First 4 weeks 18.9, 20.7, 22.2, 23.3

   Let L be the limit of (1) L = 0.78L + 6 => L = 20 ÷ 0.22 = 27.27 tons

Compound Interest & Depreciation

In each example set up a recurrence relation and a formula for A_n

in terms of n where n = number of years

A_n = amount (or value) after n years

1. Compound Interest

Principal Rate Time

(p.a.) (years)

a) £400 8% 6

b) £730 5% 8

c) £640 9% 12

d) £2400 41/2% 7

e) £1850 3% 6

2. Depreciation

a) A car costs £9350 new. It depreciates by 18% pa. Find its value after 6 years.

b) A tractor costs £22450 new. It depreciates by 23% pa. Find its value after 5 years.

c) A rabbit population is falling on average by 7% pa. If there were approximately 28500 in 1987, how many are estimated for 1995 ?

d) A caravan costs £16480 new. It depreciates annually by 13%.

Find its value after 5 years.

Compound Interest & Depreciation Solutions

1. a) $A_{n+1} = 1.08A_n$ $A_n = 1.08^n.A_0$ $A_6 = 1.08^6 . 400$ = £634.75

b) $A_{n+1} = 1.05A_n$ $A_n = 1.05^n.A_0$ $A_8 = 1.08^8 . 730$ = £1078.54

c) $A_{n+1} = 1.09A_n$ $A_n = 1.09^n.A_0$ $A_{12} = 1.09^{12} . 640$ = £1800.11

d) $A_{n+1} = 1.045A_n$ $A_n = 1.045^n.A_0$ $A_7 = 1.045^7 . 2400$ = £3266.07

e) $A_{n+1} = 1.03A_n$ $A_n = 1.03^n.A_0$ $A_6 = 1.06^6 . 1850$ = £2209.00

2. a) $A_{n+1} = 0.82A_n$ $A_n = 0.82^n.A_0$ $A_6 = 0.82^6 . 9350$ = £2842.46

b) $A_{n+1} = 0.77A_n$ $A_n = 0.77^n.A_0$ $A_5 = 0.77^5 . 22450$ = £6076.73

c) $A_{n+1} = 0.93A_n$ $A_n = 0.93^n.A_0$ $A_8 = 0.93^8 . 28500$ = 15948 rabbits

d) $A_{n+1} = 0.87A_n$ $A_n = 0.87^n.A_0$ $A_5 = 0.87^5 . 16480$ = £8213.98

Revision Exercises

1. A loch contains 48 tons of pollution. Each week the Council remove 37% of it but at the end of the week a local factory is allowed to dump 2.5 tons of rubbish into the loch. Set up a recurrence relation to describe the state of the loch and find the amount of pollution in the loch during the first four weeks. Find the long term state of the loch.

2. An initial dose of 60mg of a drug is given to a patient. The decay factor of the drug is 0.75 every 20 minutes (i.e. 75% of the drug remains in the bloodstream every 20 minute period). If a dose of 59mg is given on the hour and each hour thereafter, find the long term state of the drug in the body.

3. The sum of n terms of a sequence is given by

$$S_n = n(3n + 2)$$

a) Find the first four terms

b) Find a recurrence relation for the sequence and a formula for u_n in terms

of n.

4. Write down a recurrence relation for each of the following sequences and hence find a formula for u_n

a) 13, 21, 29, 37, . .

b) 100, 25, 6.25, . . .

c) 3, 4.25, 5.50, 6.75, . . .

d) 50, 45, 40, 35, . . .

5. The monks of St. Leonard's Priory distil their own whisky. The whisky is matured for 10 years in oak casks, during which time it loses 4% of its volume each year due to evaporation.

a) The monks prepare 7000 litres of spirit estimating that this will produce at least 5000 litres of whisky in 10 years time. Are they right ?

b) Calculate the least whole number of litres which would produce 5000 litres of whisky in 10 years time.

c) At another priory the monks have a different approach. They make the spirit in 100 litre casks and top up each year with 3 litres of additional spirit. If this process goes on, of 4% loss and 3 litres top up, predict the volume left in each cask after 500 years.

6. A mushroom bed contains some 3000 mushrooms. Each morning 30% of the mushrooms are picked. By the end of the day some 400 new mushrooms become ready for picking.

a) How many mushrooms will there be in the bed at the end of one week ?

b) Estimate how many mushrooms there will be in the bed after one year.

7. A neutron can transform into a proton and an electron. During an experiment approximately 8% of the neutrons change every minute in the way described.

a) If initially there were 500000 neutrons, how many will be left after

i) 1 minute ii) 2 minutes iii) 3 minutes

b) It is possible to model this situation using a recurrence relation of the form

$u_n = a.u_{n-1}$ where a is constant. State the value of a.

c) How long will the experiment have to run until there are fewer than 100000 neutrons left ? (Answer to the nearest minute).

8. Repeat question 2 for initial dose 50mg, decay factor 0.80 every 15 minutes with 40mg doseo n the hour every hour.

Revision Exs. Solutions

1. $A_0 = 48$ v ; $A_1 = (A_0 - 37\%$ of $A_0) + 25 = 0.63A_0 + 2.5$

   $A_{n+1} = 0.63A_n + 2.5$ (1) gives 48, 32.74, 23.13, 17.07, 13.25

   Let L be the limit of (1) => $L = 0.63L + 2.5$ => $0.37L = 2.5$ => $L = 6.76$ tons

2. $A_0 = 60$ $A_1 = 45$ $A_2 = 33.75$ $A_3 = 25.3125 + 50 = 75.3$ {1 hour}

   $A_4 = 56.48$ $A_5 = 42.36$ $A_6 = 31.77 + 50 = 81.77$ {2 hours}

   $A_7 = 61.33$ $A_8 = 46.00$ $A_9 = 34.45 + 50 = 84.45$

   $A_{10} = 63.34$ $A_{11} = 47.50$ $A_{12} = 35.63 + 50 = 85.63$ etc.

   Let $B_0 = 60$ factor $0.75^3 = 0.421875$ $B_1 = 0.421875B_0 + 50$

   $B_{n+1} = 0.421875B_n + 50$ (1) gives 75.32, 81.77, 84.50, 85.65, 86.13,

   Let L be the limit of (1) => $L = 0.421875L + 50$

   => $0.578125L = 50$ => $L = 86.49$ {to 2 dp}

3. $S_n = n(3n + 2)$ $S_1 = 5$ $S_2 = 16$ $S_3 = 33$ $S_4 = 56$

   $u_1 = 5$ $u_2 = 11$ $u_3 = 17$ $u_4 = 23$

   $u_{n+1} = u_n + 6$, $u_1 = 5$; $u_n = 6n - 1$

4. a) $u_{n+1} = u_n + 8$, $u_1 = 13$ $u_n = 8n + 5$

   b) $u_{n+1} = 1/4;$ u_n , $u_1 = 100$ $u_n = 0.25^n . 100$

   c) $u_{n+1} = u_n + 1.25$, $u_1 = 3$ $u_n = 1.25n + 1.75$

   d) $u_{n+1} = u_n - 5$, $u_1 = 50$ $u_n = 55 - 5n$

5. a) $A_0 = 7000$ $A_1 = A_0$ -30% of $A_0 = 0.96A_0$ $A_n = 0.96^nA_0$

   $A_{10} = 4653.8$ litres

   b) $0.96^{10} . x = 5000$ => $x = 7521$ litres

   c) $A_0 = 100$ $A_1 = (A_0 - 4\%$ of $A_0) + 3$ $A_{n+1} = 0.96A_n + 3$ (1)

   Let L be the limit of (1) => $L = 0.96L + 3$ => $0.04L = 3$ => $L = 75$ litres

6. $A_0 = 3000$ $A_1 = (A_0 - 30\%$ of $A_0) + 400 = 0.70A_0 + 400$

   $A_{n+1} = 0.70A_n + 400$ (1) with $A_0 = 3000$ gives 3000, 2500, 2150, 1905, 1734, 1613, 1529, 1471, . . .

Let L be the limit of (1) => $L = 0.7L + 400$ => $0.L = 400$ => $L = 1333$

7. $A_0 = 500000$ $A_1 = A_0 - 8\%$ of $A_0 = 0.92A_0$ $A_n = 0.92^n A_0$

   a) $A_1 = 460000$ $A_2 = 423200$ $A_3 = 389344$

   b) $a = 0.92$

   c) $0.92^n \, . \, 500000 < 100000$ => $0.92^n < 0.2$ => $n \ln 0.92 < \ln 0.2$

   => $n > \ln 0.2$ ÷ $\ln 0.92$ => $n > 19.3$ => $n = 20$

8. $A_0 = 50$

After 15mins $A_1 = 40$

After 30 mins $A_2 = 32$

After 45 mins $A_3 = 25.6$

After 1 hour $A_4 = 20.48 + 40 = 60.48$

After 1 hour 15mins $A_5 = 48.83$

After 1 hour 30 mins $A_6 = 38.707$

After 1 hour 45 mins $A_7 = 30.966$

After 2 hours $A_8 = 24.77 + 40 = 64.77$

After 2 hours 15mins $A_9 = 51.82$

After 2 hours 30 mins $A_{10} = 41.45$

After 2 hours 45 mins $A_{11} = 33.16$

After 3 hours $A_{12} = 26.53 + 40 = 66.53$

After 3 hours 15mins $A_{13} = 53.22$

After 3 hours 30 mins $A_{14} = 42.58$

After 3 hours 45 mins $A_{15} = 34.06$

After 4 hours $A_{16} = 27.25 + 40 = 67.25$

After 4 hours 15mins $A_{17} = 53.80$ etc.

Consider what is happening each hour on the hour.

$B_0 = 50$ $0.8^4 = 0.4096$

$B_1 = 0.4096B_0 + 40$

$B_2 = 0.4096B_1 + 40$

etc.

$B_{n+1} = 0.4096B_n + 40$ (1) with $B_0 = 50$ gives 60.48, 64.77, 66.53, 67.25, . .

tending to the limit of 67.75 {using the calculator}

Let L be the limit of (1) => $L = 0.4090L + 40$

=> $0.5904L = 40$

=> $L = 40$ ÷ $0.5904 = 67.\ 75$